Some remarks on the cohomology of an invertible sheaf

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ABSTRACT

This note is mainly devoted to three-folds. In the first part we give an example of an invertible sheaf $\mathscr{L}$ with Kodaira dimension three such that the base locus of the complete linear system $|\mathscr{L}|$ has codimension two and $H^2(\mathscr{L}^{-1}) \neq (0)$. This example gives a negative answer to a question posed by C. P. Ramanujam in [*R*]. Moreover we obtain a result on the vanishing of the second cohomology group.

In the second part we extend to the case of dimension greater than two same results proved by Zariski in [*Z*2] for surfaces. These results concern the cohomology of high multiples of an effective divisor.

1. AN EXAMPLE ON THE VANISHING OF THE SECOND COHOMOLOGY GROUP

Hereinafter we denote by X a non singular, projective variety over the complex field. For every invertible sheaf $\mathscr{F}$ on X we write $|\mathscr{F}|$ for the complete linear system determinated by $H^0(X, \mathscr{F})$ and we denote $h^i(\mathscr{F})$ the dimension of $H^i(X, \mathscr{F})$. Moreover, we write $\mathscr{F}_Y$ and $\mathscr{F}^n$ for $\mathscr{F} \otimes \mathscr{O}_Y$ and $\mathscr{F}^{\otimes n}$ respectively, Y being a subvariety of X, $n \in \mathbf{N}$. Finally, we note by $\varkappa(\mathscr{F})$ the Kodaira dimension of $\mathscr{F}$.

C. P. Ramanujam proves in [R], Th. 3, using a result of Grauert-Riemenschneider, the following generalization of the Kodaira vanishing theorem for algebraic varieties.

THEOREM 1.1. *Let $\mathscr{L}$ be an invertible sheaf on a variety X of dimension n, m an integer with $1 < m < n$ such that for some $N > 0$ the complete linear* *system* $|\mathscr{L}^N|$ *has the dimension of its base point set* $\leqslant n-m$ *and has a projective image of dimension* $\geqslant m$. *Then* $H^i(X, \mathscr{L}^{-1})=(0)$ *for* $0\leqslant i<m$.

Next, the Author observes that it is not clear whether the hypothesis on the base point set of $|\mathscr{L}^N|$ can be weakened to the assumption that the base point set be of dimension $\leqslant n-2$.*

The following example gives a negative answer to the previous question.

EXAMPLE. Take in $X=\mathbf{P}^3(\mathbf{C})$ a non singular, elliptic curve C of order ν. Denoting by $\mathscr{L}$ the sheaf $\mathcal{O}_X(n)$, n fixed positive integer, we consider on C a divisor $D=\sum_{i=1}^{n\nu} P_i$ and the sheaf $\mathcal{O}_C(\eta)=\mathscr{L}\otimes\mathcal{O}_C(-D)$. Since $\mathrm{Pic}^0(C)\neq(0)$ it is possible to choose D so that $\mathcal{O}_C(\eta)$ is not isomorphic to $\mathcal{O}_C$.

Let $\tau\colon\tilde{X}\to X$ be the blowup of X along D. We denote by E_i the exceptional divisors, $i=1,\ldots,n\nu$, by $\tilde{C}$ the proper transform of C and by $\tilde{\mathscr{L}}$ the sheaf $\tau^*\mathscr{L}\otimes\mathcal{O}_{\tilde{X}}(-\sum_{i=1}^{n\nu}E_i)$. Then $\tilde{\mathscr{L}}_{\tilde{C}}\simeq\tau^*\mathcal{O}_C(\eta)$, hence $\deg\tilde{\mathscr{L}}_{\tilde{C}}=0$. Moreover, since the curves $\tilde{C}$, C are isomorphic, one has $\tilde{\mathscr{L}}_{\tilde{C}}\not\simeq\mathcal{O}_{\tilde{C}}$. Therefore $H^0(\tilde{\mathscr{L}}_{\tilde{C}})=(0)$. It follows that $\tilde{C}$ is a base curve of $|\tilde{\mathscr{L}}|$.

Furthermore, let $\sigma\colon X'\to\tilde{X}$ be the blowup of $\tilde{X}$ in a point P belonging to $\tilde{C}$. Put $\tilde{\mathscr{L}}=L$ and consider on X' the exact cohomology sequence:

$$(1)\qquad 0\to H^0(L')\to H^0(\sigma^*L)\to H^0(\sigma^*L\otimes\mathcal{O}_{E'})\to H^1(L'),$$

where $E'\simeq\mathbf{P}^2$ is the exceptional divisor of σ and $L'=\sigma^*L\otimes\mathcal{O}_{X'}(-E')$. Now, we have $\sigma^*L\otimes\mathcal{O}_{E'}\simeq\mathcal{O}_{E'}$ hence $H^0(\sigma^*L\otimes\mathcal{O}_{E'})\neq(0)$. Moreover E' is a base component of $|\sigma^*L|$ since $\tilde{C}$ is a base curve of $|L|$. It follows $H^0(L')\simeq H^0(\sigma^*L)$. Therefore, recalling (1), we have $H^1(L')\neq(0)$, hence $H^2(\omega_{X'}\otimes L'^{-1})\neq(0)$ by duality. On the other hand $\varkappa(\omega_{X'}^{-1}\otimes L')=3$. Indeed, putting $\varphi=\tau\circ\sigma$ and denoting by E_i^*, $i=1,\ldots,n\nu$, the inverse images in X' of the exceptional divisors E_i of $\tilde{X}$, one has:

$$\begin{aligned}\omega_{X'}^{-1}\otimes L' &\simeq \varphi^*\mathcal{O}_X(4)\otimes\mathcal{O}_{X'}(-2(E'+\sum_{i=1}^{n\nu}E_i^*))\otimes L'\\ &\simeq \varphi^*\mathcal{O}_X(n+4)\otimes\mathcal{O}_{X'}(-3(E'+\sum_{i=1}^{n\nu}E_i^*)).\end{aligned}$$

Choosing now the integer n such that $\varkappa(\mathcal{O}_X(n+4)\otimes\mathscr{I}_{3C})=3$ we are done. Finally, it is easy to see that the proper transform C' of $\tilde{C}$ is the base locus of $|\omega_{X'}^{-1}\otimes L'|$ and $\deg(\omega_{X'}^{-1}\otimes L'\otimes\mathcal{O}_{C'})<0$ for large n.

This example suggest the following statement on the vanishing of the second cohomology group.

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* Let $\mathscr{F}$ be an invertible sheaf and let $\mathscr{F}'$ be the subsheaf of $\mathscr{F}$ generated by $\mathrm{H}^0(X,\mathscr{F})$. Then there is a unique sheaf of ideals I such that $\mathscr{F}'\simeq\mathrm{I}\otimes\mathscr{F}$. We define the *base point set* of $|\mathscr{F}|$ to be the support of the subscheme defined by I.

PROPOSITION 1.2. *Let X be a variety of dimension three, $\mathscr{L}$ an invertible sheaf with $\varkappa(\mathscr{L})=3$ and such that the base point set of $|\mathscr{L}|$ is a non singular curve C. Then if* $\deg \mathscr{L}_C=0$ *and $\mathscr{I}_C/\mathscr{I}_C^2$ is ample* we have* $H^2(X, \mathscr{L}^{-1})=(0)$.

For the proof we need the following Lemma. Let $\sigma\colon \tilde{X} \to X$ be the blowup of X along C, $\tilde{\mathscr{L}}$ the proper transform of $\mathscr{L}$, E the exceptional divisor.

LEMMA 1.3. *Suppose the sheaf $\mathscr{L}$ to verify the hypotheses of Proposition 1.2. Then there exists an integer $q \gg 0$ such that $\tilde{\mathscr{L}}^q$ is generated by its global sections.*

PROOF. Assume $\tilde{\mathscr{L}}$ to have a base curve $\mathscr{E}=\bigcup_{i=1}^r \mathscr{E}_i$. Obviously $\mathscr{E}$ is contained in E. On the other hand the hypotheses made ensure the ampleness of the sheaf $\tilde{\mathscr{L}}_E=\sigma^*\mathscr{L}\otimes \mathcal{O}_{\tilde{X}}(-sE)\otimes \mathcal{O}_E$, s denoting the multiplicity of $\mathscr{L}$ along C. Hence $\deg \tilde{\mathscr{L}}_{\mathscr{E}_i}>0$, $i=1, \ldots, r$. We prove that this leads to a contradiction. By Bertini's Theorem there exists a surface $S\in|\tilde{\mathscr{L}}^q|$, $q\gg 0$, S reduced and irreducible. Moreover we have

$$\tilde{\mathscr{L}}_S^q \simeq \mathbf{L}\otimes \mathcal{O}_S\Big(\sum_{i=1}^r \alpha_i\mathscr{E}_i\Big)$$

where $|\mathbf{L}|$ has not fixed components and the α_i's are positive integers. Consider on S the sheafs:

$$\theta_{0,0}^{(n)}=\tilde{\mathscr{L}}_S^{n-q},\ \theta_{i,s_i}^{(n)}=\tilde{\mathscr{L}}_S^n\otimes \mathscr{I}_{\mathscr{E}_i}^{\alpha_i-s_i}\otimes \mathscr{I}_{\sum_{j>i}\alpha_j\mathscr{E}_j},\quad i=1, \ldots, r,\ s_i=0, \ldots, \alpha_i,$$

n positive integer. One has:

$$\theta_{0,0}^{(n)}\subset\theta_{1,0}^{(n)},\ \theta_{i,\alpha_i}^{(n)}\simeq\theta_{i+1,0}^{(n)},\quad i=1, \ldots, r-1,$$

$$\theta_{i,s_i-1}^{(n)}\subset\theta_{i,s_i}^{(n)},\quad i=1, \ldots, r,\ s_i\neq 0.$$

Since $\deg \tilde{\mathscr{L}}_{\mathscr{E}_i}>0$ the sheafs $\tilde{\mathscr{L}}_{\mathscr{E}_i}$, $i=1, \ldots, r$, are amples. Hence:

$$h^1(\theta_{i,s_i}^{(n)}/\theta_{i,s_i-1}^{(n)})=0,\quad i=1, \ldots, r,\quad s_i\neq 0,\quad n\gg 0.$$

Moreover, if $\mathbf{L}=\mathcal{O}_S(B)$, we have $\theta_{1,0}^{(n)}/\theta_{0,0}^{(n)}\simeq(\tilde{\mathscr{L}}_S^n\otimes \mathscr{I}_{\sum\alpha_i\mathscr{E}_i})\otimes \mathcal{O}_B$. On the other hand $\deg \tilde{\mathscr{L}}_B\geqslant 0$, then $h^1(\theta_{1,0}^{(n)}/\theta_{0,0}^{(n)})=$ constant $c\geqslant 0$ for large n. It follows:

$$h^1(\theta_{0,0}^{(n)})+c\geqslant h^1(\theta_{1,0}^{(n)})\geqslant \ldots \geqslant h^1(\theta_{r,\alpha_r-1}^{(n)}),\ \text{i.e.}$$

$$h^1(\tilde{\mathscr{L}}_S^{n-q})+c\geqslant h^1(\tilde{\mathscr{L}}_S^n\otimes \mathscr{I}_{\mathscr{E}_r}),\quad n\gg 0. \tag{2}$$

Now we consider the exact sequence:

$$0\to \tilde{\mathscr{L}}_S^n\otimes \mathscr{I}_{\mathscr{E}_r}\to \tilde{\mathscr{L}}_S^n\to \tilde{\mathscr{L}}_{\mathscr{E}_r}^n\to 0.$$

* In the sense of [H], p. 83.

Since $\tilde{\mathscr{L}}_{\mathscr{E}_r}$ is ample one has $H^1(\tilde{\mathscr{L}}^n_{\mathscr{E}_r})=(0)$, $n \gg 0$. Using the inequality (2) we get the cohomology sequence:

$$(3) \qquad 0 \to H^0(\tilde{\mathscr{L}}^n_S \otimes \mathscr{I}_{\mathscr{E}_r}) \to H^0(\tilde{\mathscr{L}}^n_S) \to H^0(\tilde{\mathscr{L}}^n_{\mathscr{E}_r}) \to \mathbb{C}^N \to 0,$$

with N constant for large n.

Let $\varrho^{(n)}$ the superabundance of $|\tilde{\mathscr{L}}^n_S|$, i.e.

$$\varrho^{(n)} = h^0(\tilde{\mathscr{L}}^n_S) - \dim. \operatorname{Im} \{H^0(\tilde{\mathscr{L}}^n) \xrightarrow{\psi} H^0(\tilde{\mathscr{L}}^n_S)\}.$$

Since $\mathscr{E}_r$ is a base curve it follows $\operatorname{Im} \psi \subset H^0(\tilde{\mathscr{L}}^n_S \otimes \mathscr{I}_{\mathscr{E}_r})$, hence

$$\varrho^{(n)} \geqslant h^0(\tilde{\mathscr{L}}^n_S) - h^0(\tilde{\mathscr{L}}^n_S \otimes \mathscr{I}_{\mathscr{E}_r}).$$

Therefore, recalling (3), $\varrho^{(n)}$ increases with n as $h^0(\tilde{\mathscr{L}}^n_{\mathscr{E}_r})$. So we get a contradiction: it suffices to look at the exact cohomology sequence deduced from:

$$0 \to \tilde{\mathscr{L}}^{n-q} \to \tilde{\mathscr{L}}^n \to \tilde{\mathscr{L}}^n_S \to 0,$$

and to observe that $h^1(\tilde{\mathscr{L}}^n_S)=$ constant for large n (see Proposition 2.1 and also [Z2], 6.4.(a)).

PROOF OF PROPOSITION 1.2. We put $\mathscr{F} = \tilde{\mathscr{L}}^{-1} \otimes \mathscr{O}_{\tilde{X}}(-E)$. By Theorem 1.1. and Lemma 1.3. we have $H^1(\tilde{\mathscr{L}}^{-1}_E) = H^2(\tilde{\mathscr{L}}^{-1}) = (0)$. Then, from the exacteness of

$$H^1(\tilde{\mathscr{L}}^{-1}_E) \to H^2(\mathscr{F}) \to H^2(\tilde{\mathscr{L}}^{-1}),$$

we obtain $H^2(\mathscr{F})=(0)$. Therefore the exact sequence

$$0 \to \mathscr{F} \otimes \mathscr{O}_{\tilde{X}}(-E) \to \mathscr{F} \to \mathscr{F}_E \to 0$$

gives a surjective morphism:

$$H^1(\mathscr{F}_E) \to H^2(\mathscr{F} \otimes \mathscr{O}_{\tilde{X}}(-E)) \to 0.$$

But the hypotheses made ensure the ampleness of $\mathscr{F}^{-1}_E$, hence $H^1(\mathscr{F}_E)=(0)$ and therefore $H^2(\mathscr{F} \otimes \mathscr{O}_{\tilde{X}}(-E))=(0)$. Repeating the same reasoning we obtain $H^2(\mathscr{F} \otimes \mathscr{O}_{\tilde{X}}(-(s-1)E))=(0)$, i.e. $H^2(\sigma^*\mathscr{L}^{-1})=(0)$. But Leray's spectral sequence gives $H^2(\sigma^*\mathscr{L}^{-1}) \simeq H^2(\mathscr{L}^{-1})$, so we are done.

It is possible to generalize Proposition 1.2. as follows:

THEOREM 1.4. *Let $\mathscr{L}$ be an invertible sheaf on a variety X of dimension n, such that $\varkappa(\mathscr{L})>2$. Suppose the base point set B of $|\mathscr{L}|$ to be non singular and* $\dim. B \leqslant n-2$. *Then, if $\mathscr{L} \otimes \mathscr{O}_B \sim \mathscr{O}_B$* and $\mathscr{I}_B/\mathscr{I}^2_B$ is ample we have* $H^2(X, \mathscr{L}^{-1})=(0)$.

* $\sim$ means numerical equivalence.

PROOF. Choose a section H of X obtained by a general hypersurface of order r and consider the exact sequence:

$$0 \to \mathscr{L}^{-1} \otimes \mathcal{O}_X(-H) \to \mathscr{L}^{-1} \to \mathscr{L}_H^{-1} \to 0.$$

Since $H^2(\mathscr{L}^{-1} \otimes \mathcal{O}_X(-H)) = (0)$ for large r (see [F], p. 269) the assertion immediately follows by induction on n and taking into account that the hypotheses made are preserved on H.

We conclude this part proving a converse implication with respect to Theorem 1.1.

LEMMA 1.5. *Let $\mathscr{L}$ be an invertible sheaf on a variety X of dimension $n \geqslant 2$. Suppose that $\mathscr{L}^q$ is generated by its global sections for some integer $q > 0$. Therefore for every $i = 1, \ldots, n-1$ we have:*

$$H^j(\mathscr{L}^{-m}) = (0), \ m \geqslant 1, \ j = 0, \ldots, i \Leftrightarrow \varkappa(\mathscr{L}) > i.$$

PROOF. By Theorem 1.1. it remains to show the implication $\Rightarrow$. In the case $i = 1$ the assertion easily follows using Leray's spectral sequence (see [M]). Therefore, reasoning by induction on i, it suffices to prove that if $H^j(\mathscr{L}^{-m}) = (0)$, $j = 0, \ldots, i$, $m \geqslant 1$, the equality $\varkappa(\mathscr{L}) = i$ is excluded.

We put $\mathscr{L} = \mathcal{O}_X(\mathscr{D})$. By Bertini's Theorem there exists a divisor $D \in |q\mathscr{D}|$, D irreducible and non singular. On the other hand the exactness of

$$H^j(\mathscr{L}^{-m}) \to H^j(\mathscr{L}_D^{-m}) \to H^{j+1}(\mathscr{L}^{-m-q})$$

gives $H^j(\mathscr{L}_D^{-m}) = (0)$ for every $j = 0, \ldots, i-1$, $m \geqslant 1$. Thus by induction hypothesis, denoting by φ the morphism associated to $\mathscr{L}^q$, we have $\dim. \varphi(D) > i-1$, hence $\varkappa(\mathscr{L}) > i$, otherwise it would be $\dim. \varphi(D) = i-1$, a contradiction.

2. ON THE SUPERABUNDANCE OF A LINEAR SYSTEM

The results to follow hold for an algebraically closed field k of arbitrary characteristic. We begin proving a Proposition which extends Theorem 6.4(a) of [Z2].

PROPOSITION 2.1. *Let $\mathscr{L} = \mathcal{O}_X(\mathscr{D})$ be an invertible sheaf on a (non singular) variety X of dimension $d \geqslant 2$ such that $\mathscr{D} \succ 0$ and $\varkappa(\mathscr{L}) > 1$. Moreover, suppose that $|\mathscr{L}|$ has not fixed components. Then, for every invertible sheaf $\mathscr{F} = \mathcal{O}_X(\Delta)$, we have:*

$$h^{d-1}(\mathscr{F} \otimes \mathscr{L}^n) = \textit{constant}, \textit{ for large } n.$$

PROOF. By Bertini's Theorem there exists a divisor $D \in |\mathscr{D}|$, D reduced and irreducible. We consider the exact sequence

$$0 \to \mathscr{F} \otimes \mathscr{L}^{n-1} \to \mathscr{F} \otimes \mathscr{L}^n \to \mathscr{F} \otimes \mathscr{L}_D^n \to 0.$$

It suffices to show that $h^{d-1}(\mathscr{F} \otimes \mathscr{L}^{n-1}) \geqslant h^{d-1}(\mathscr{F} \otimes \mathscr{L}^{n})$ or, equivalently, $H^{d-1}(\mathscr{F} \otimes \mathscr{L}_D^n) = (0)$, for $n \gg 0$. By duality we have on D:

$$H^{d-1}(\mathscr{F} \otimes \mathscr{L}_D^n) \simeq H^0(\omega_D \otimes (\mathscr{F} \otimes \mathscr{L}_D^n)^{-1}) \simeq H^0(\mathscr{O}_D(K_D - D\cdot\Delta - nD^2)).*$$

In the case $d=2$ the assert immediately follows since

$$\deg(K_D - D\cdot\Delta - nD^2) < 0$$

if n is large enough. Now suppose $d \geqslant 3$ and let H be a generic hyperplane section. Let us assume that there exists a divisor $L_n \in |K_D - D\cdot\Delta - nD^2|$. Then we have:

$$(H^{d-2}\cdot L_n) = (H^{d-2}\cdot K_D) - (H^{d-2}\cdot D\cdot\Delta) - n(H^{d-2}\cdot D^2).$$

By use of Nakai Criterion we are done. In fact one has $(H^{d-2}\cdot D^2) > 0$ since H is ample and $\alpha D^2 \succ 0$ for $\varkappa(\mathscr{L}) > 1$, with α suitable positive integer. It follows $(H^{d-2}\cdot L_n) < 0$ for large n, a contradiction. Therefore $|K_D - D\cdot\Delta - nD^2| = \emptyset$ for $n \gg 0$.

REMARK. According to the proof, Proposition 2.1. can be reformulated, more precisely, as follows: *there exists a positive integer m_0 such that $h^{d-1}(\mathscr{F} \otimes \mathscr{L}^m) \leqslant h^{d-1}(\mathscr{F} \otimes \mathscr{L}^{m_0})$ for every $m > m_0$; furthermore $m_0 = [n_0]$ where $n_0 = (H^{d-2}\cdot K_D - H^{d-2}\cdot D\cdot\Delta)/(H^{d-2}\cdot D^2)$.*

In particular, in the case $\mathscr{F} = \mathscr{O}_X$, $\mathscr{L} = \omega_X$ we have $n_0 = 2$ and therefore $h^{d-1}(\omega_X^n) \leqslant h^{d-1}(\omega_X^2)$ for $n \geqslant 2$.

COROLLARY 2.2. (a) *Let us assume that $|\mathscr{L}^q|$ has no fixed components for some integer $q \geqslant 1$. Then $h^{d-1}(\mathscr{L}^m)$ is a periodic function of m, for large m, with period q.*

(b) *If $|\mathscr{L}^q|$ has no fixed components for every $q \geqslant q_0$, q_0 fixed integer, then $h^{d-1}(\mathscr{L}^n) = $ constant for large n.*

PROOF. (a) By Proposition 2.1. we have $h^{d-1}(\mathscr{L}^{mq+i}) = c_i$ (constant) for every $i = 0, \dots, q$, $n \gg 0$. Thus $h^{d-1}(\mathscr{L}^m) = c_i$ whenever $m \equiv i$ (mod. q). Therefore $h^{d-1}(\mathscr{L}^m) = h^{d-1}(\mathscr{L}^{m+q})$ for $m \gg 0$.

(b) One has $h^{d-1}(\mathscr{L}^m) = h^{d-1}(\mathscr{L}^{m+q_0+r})$ for large m and $r \geqslant 0$. Hence $H^{d-1}(\mathscr{L}^n) = $ constant, $n \gg 0$.

Next we consider the case of three-folds. We shall prove the following:

PROPOSITION 2.3. *Let X be a variety of dimension 3 and let $\mathscr{L} = \mathscr{O}_X(\mathscr{D})$ be an invertible sheaf on X, such that $\mathscr{D} \succ 0$, $\varkappa(\mathscr{L}) = 3$. Assume that $\mathscr{L}$ is generated by its global sections and let $\varphi: X \to Y$ be the morphism defined*

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* Here K_D denotes the divisor associated to the dualizing sheaf ω_D.

by $\mathscr{L}$. Then, if $\delta(\mathscr{L}^n)$ is the superabundance of $|\mathscr{L}^n|$, the equality

$$\delta(\mathscr{L}^n)=p_a(X)-p_a(Y)+n(p_a(C)-p_a(D))^{*_1}$$

holds for large n, where $D\in|\mathscr{D}|$ and $C=\varphi^*(D)$.[*2]

PROOF. Replacing $\mathscr{L}$ by some $\mathscr{L}^q$, q positive integer, we can suppose φ birational and Y normal (see [Z2], § 6). As before, there exists a divisor $D\in|\mathscr{D}|$, D reduced and irreducible (in the case ch. $k=0$ we can also suppose D non singular).

Obviously $h^0(\mathscr{O}_X(nD))=h^0(\mathscr{O}_Y(nC))$ for every $n>0$. Putting for semplicity $nC=C_n$ we have (see [Z1], p. 584):

$$h^0(\mathscr{O}_Y(nC))=-p_a(Y)-p_a(-C_n),$$

where

$$p_a(-C_n)=-4+\sum_{i=1}^{3}(-1)^i\,p_a(C_n^i).$$

Whence:

$$h^0(\mathscr{O}_X(nD))=-p_a(Y)-\sum_{i=1}^{3}(-1)^i\,p_a(C_n^i)+4. \tag{4}$$

The Riemann-Roch Theorem gives:

$$h^0(\mathscr{O}_X(nD))=(nD)^3-p_a(X)+p_a(nD)-p_a((nD)^2)+i(nD)+\delta(\mathscr{L}^n)+3,$$

where $i(nD)=h^0(\omega_X\otimes\mathscr{O}_X(-nD))$. Using Nakai's Criterion one has $i(nD)=0$ for $n\gg 0$ since $D\succ 0$. Therefore, recalling (4):

$$\delta(\mathscr{L}^n)=-p_a(Y)-\sum_{i=1}^{3}(-1)^i\,p_a(C_n^i)+4-(nD)^3+$$
$$+p_a(X)-p_a(nD)+p_a((nD)^2)-3.$$

On the other hand $p_a(C_n^3)=(C_n^3)-1$ (see [Z1] Part III, § 10). Moreover, let U be the open subset on which the restriction $\varphi_{\varphi^{-1}(U)}$ is an isomorphism and put $Z=Y\setminus U$. Taking into account that φ has connected fibres and X is irreducible (see also [EGA] III, 4.3.2, 4.3.12) we have $\mathrm{cd}_Y Z\geqslant 2$. Hence:

$$\begin{aligned}&(C_n^3)=(nD)^3 \text{ (and therefore } p_a(C_n^3)=(nD)^3-1),\\ &p_a(C_n^2)=p_a((nD)^2).\end{aligned} \tag{5}$$

In fact, in the case dim. $Z=0$ we can choose D such that the restriction $\varphi_D\colon D\to C$ is an isomorphism, so formulas (5) are trivial. Otherwise,

[*1] We denote by $p_a(V)$ the arithmetic genus of a variety V, i.e. $(-1)^{\dim V}\,p_a(V)=$ $=\chi(\mathscr{O}_V)-1$.

[*2] The fact that $\delta(\mathscr{L}^n)$ is, for large n, a polinomial of n of degree 1 is also a consequence of a more general result proved in [L-S].

if dim. $Z=1$, we look at the curve C_n^2 of C. We can assume that C_n^2 is a multiple of a generic hyperplane section of C and does not contain points of $Z \cap C$. Then $p_a(C_n^2)=p_a((nD)^2)$. Analogously we have $(C_n^3)=(nD)^3$. Finally we get:

$$p_a(C_n)-p_a(nD)=n(p_a(C)-p_a(D)). \tag{6}$$

In fact it results (see also [S], § 4):

$$p_a(C_n)=np_a(C)+\binom{n}{2}p_a(C^2)+\binom{n}{3}p_a(C^3),$$
$$p_a(nD)=np_a(D)+\binom{n}{2}p_a(D^2)+\binom{n}{3}(D^3).$$

Hence we have (6) by use of (5). In conclusion:

$$\delta(\mathscr{L}^n)=p_a(X)-p_a(Y)+n(p_a(C)-p_a(D)).$$

REMARK. In the case ch. $k=0$ the equality $p_a(C)-p_a(D)=h^1(\mathcal{O}_D((nD)^2))$ holds for large n. Indeed, it is easy to see that the curve $\gamma=D^2$ of the non singular surface D is such that $|\gamma|$ is base point free and $\gamma^2>0$. Then the assertion follows by [Z2], 6.3, 6.5(b).

As a consequence we have the equivalence: $h^1(\mathscr{L}^n)=$ *constant for large* n *iff* $p_a(C)=p_a(D)$. In fact, consider the exact sequence:

$$H^1(\mathscr{L}^{n-1}) \to H^1(\mathscr{L}^n) \to H^1(\mathscr{L}_D^n).$$

Since $h^1(\mathscr{L}_D^n)=p_a(D)-p_a(C)$ for large n, one has $h^1(\mathscr{L}^{n-1})\geqslant h^1(\mathscr{L}^n)$, $n \gg 0$. The conclusion immediately follows from Proposition 2.3.

COROLLARY 2.4. *Suppose that the canonical sheaf* ω_X *of* X *satisfies the hypotheses of* 2.3. *Then* $p_a(X)=p_a(Y)$.

PROOF. Theorem 1.1. can be applied to ω_X. Hence, by duality, we have $\delta(\omega_X^n)=0$ for every $n\geqslant 2$, i.e. $p_a(X)-p_a(Y)=p_a(C)-p_a(D)=0$.

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